Error Analysis

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In this document, I derive analytical functions for the error in each active region parameter using formal error propagation. It is important to note that the error formulae below assume that (i) all the variables are linearly independent, and (ii) the relative errors are small. And neither of those assumptions are true. But we are making them. Moving on to the derivations:

1 $\Phi = \sum_i |B_{z_i}| dA$

Φ is a sum of all pixels; therefore, the error in Φ is simply: $(\delta\Phi)^2 = \sum_i (\delta B_{z_i})^2 \, dA^2$.

2 $\overline{\gamma} = \frac{1}{N} \sum_i \arctan\left(\frac{B_{h_i}}{B_{z_i}}\right)$

The error in γ per pixel, (i, j), is as follows:
$(\delta\gamma)^2 = \left(\frac{\partial\gamma}{\partial B_h}\delta B_h\right)^2 + \left(\frac{\partial\gamma}{\partial B_z}\delta B_z\right)^2$.

The partial derivatives per pixel, (i, j), are as follows:
$\frac{\partial\gamma}{\partial B_h} = \frac{1}{\left[1+\left(\frac{B_h}{B_z}\right)^2\right] B_z}$ and $\frac{\partial\gamma}{\partial B_z} = -\frac{B_h}{\left[1+\left(\frac{B_h}{B_z}\right)^2\right] {B_z}^2}$.

Therefore, the error in γ per pixel, (i, j), is simply:
$(\delta\gamma)^2 = \left(\frac{1}{\left[1+\left(\frac{B_h}{B_z}\right)^2\right] B_z}\delta B_h\right)^2 + \left(\frac{B_h}{\left[1+\left(\frac{B_h}{B_z}\right)^2\right] {B_z}^2}\delta B_z\right)^2 = \left(\frac{1}{1+\left(\frac{B_h}{B_z}\right)^2}\right)^2 \left[\left(\frac{\delta B_h}{B_z}\right)^2 + \left(\frac{B_h \delta B_z}{{B_z}^2}\right)^2\right]$.

We report the mean value of γ, which means we must also determine the error in $\overline{\gamma}$:
$\overline{\gamma} = \frac{1}{N}\sum_i \gamma_i$, where N represents the number of pixels. Therefore, the error in $\overline{\gamma}$ is:

$(\delta\overline{\gamma})^2 = \frac{1}{N^2}\sum_i (\delta\gamma_i)^2$.

Additionally, the δB_h array must be determined. We can calculate δB_h per pixel, (i, j), as follows:

$(\delta B_h)^2 = \frac{(B_x \delta B_x)^2 + (B_y \delta B_y)^2}{{B_h}^2}$.

3 $\overline{|\nabla B_t|} = \frac{1}{N} \sum_i \sqrt{\left(\frac{\partial B_{t_i}}{\partial x}\right)^2 + \left(\frac{\partial B_{t_i}}{\partial y}\right)^2}$

There are two approaches one can take to calculate $\delta\,|\nabla B_t|$: calculate the error terms assuming (i) an analytical form for the derivative, e.g. $\frac{\partial B_t}{\partial y}$, or (ii) a numerical form, e.g. $\frac{\partial B_t}{\partial y}$ = `0.5*(Bt[i,j+1]`

`- Bt[i,j-1])`. I am going to go with the numerical form.

Using a simple finite difference method, the derivatives are defined as follows:
$\frac{\partial B_t}{\partial y}$ `=0.5*(Bx[i,j+1] - Bx[i,j-1])` and $\frac{\partial B_t}{\partial x}$ `=0.5*(Bt[i+1,j] - Bt[i-1,j])`.

Therefore, the gradient at one position (i, j) is defined as follows:
$|\nabla \mathtt{Bt[i,j]}| = \sqrt{0.25 * (\mathtt{Bt[i+1,j]} - \mathtt{Bt[i-1,j]})^2 + 0.25 * (\mathtt{Bt[i,j+1]} - \mathtt{Bt[i,j-1]})^2}$

The error in in $|\nabla B_t|$ is as follows:

$$(\delta\,|\nabla \mathtt{Bt[i,j]}|)^2 = \left(\frac{\partial \nabla B_t}{\partial \mathtt{Bt[i,j+1]}}\delta \mathtt{Bt[i,j+1]}\right)^2 + \left(\frac{\partial \nabla B_t}{\partial \mathtt{Bt[i,j-1]}}\delta \mathtt{Bt[i,j-1]}\right)^2 + \left(\frac{\partial \nabla B_t}{\partial \mathtt{Bt[i+1,j]}}\delta \mathtt{Bt[i+1,j]}\right)^2 + \left(\frac{\partial \nabla B_t}{\partial \mathtt{Bt[i-1,j]}}\delta \mathtt{Bt[i-1,j]}\right)^2. \tag{1}$$

Let's evaluate the first term of the equation above (we'll define this as $TERM1$):
$\left(\frac{\partial|\nabla B_t|}{\partial \mathtt{Bx[i,j+1]}}\delta \mathtt{Bx[i,j+1]}\right)^2 = \left(\frac{\mathtt{Bt[i,j+1]} - \mathtt{Bt[i,j-1]}}{4\sqrt{0.25*(\mathtt{Bt[i+1,j]}-\mathtt{Bt[i-1,j]})^2+0.25*(\mathtt{Bt[i,j+1]}-\mathtt{Bt[i,j-1]})^2}}\delta \mathtt{Bx[i,j+1]}\right)^2.$

And here are the subsequent terms. $TERM2$:
$\left(\frac{\partial|\nabla B_t|}{\partial \mathtt{Bx[i,j-1]}}\delta \mathtt{Bx[i,j-1]}\right)^2 = \left(\frac{-\mathtt{Bt[i,j+1]} + \mathtt{Bt[i,j-1]}}{4\sqrt{0.25*(\mathtt{Bt[i+1,j]}-\mathtt{Bt[i-1,j]})^2+0.25*(\mathtt{Bt[i,j+1]}-\mathtt{Bt[i,j-1]})^2}}\delta \mathtt{Bx[i,j-1]}\right)^2.$

$TERM3$:
$\left(\frac{\partial|\nabla B_t|}{\partial \mathtt{Bx[i+1,j]}}\delta \mathtt{Bx[i+1,j]}\right)^2 = \left(\frac{\mathtt{Bt[i+1,j]} - \mathtt{Bt[i-1,j]}}{4\sqrt{0.25*(\mathtt{Bt[i+1,j]}-\mathtt{Bt[i-1,j]})^2+0.25*(\mathtt{Bt[i,j+1]}-\mathtt{Bt[i,j-1]})^2}}\delta \mathtt{Bx[i+1,j]}\right)^2.$

$TERM4$:
$\left(\frac{\partial|\nabla B_t|}{\partial \mathtt{Bx[i-1,j]}}\delta \mathtt{Bx[i-1,j]}\right)^2 = \left(\frac{-\mathtt{Bt[i+1,j]} + \mathtt{Bt[i-1,j]}}{4\sqrt{0.25*(\mathtt{Bt[i+1,j]}-\mathtt{Bt[i-1,j]})^2+0.25*(\mathtt{Bt[i,j+1]}-\mathtt{Bt[i,j-1]})^2}}\delta \mathtt{Bx[i-1,j]}\right)^2.$

Therefore, the error in $|\nabla B_t|$ is as follows: $(\delta\,|\nabla B_t|)^2 = TERM1 + TERM2 + TERM3 + TERM4$.
We report the mean value of $|\nabla B_t|$, which means we must also determine the error in $\overline{|\nabla B_t|}$:

$\overline{|\nabla B_t|} = \frac{1}{N}\sum_i \nabla B_{t_i}$, where N represents the number of pixels. Therefore, the error in $\overline{|\nabla B_t|}$ is:
$\left(\delta\overline{|\nabla B_t|}\right)^2 = \frac{1}{N^2}\sum_i (\delta \nabla B_{t_i})^2.$

The δB_t array must be determined. We can calculate δB_t per pixel, (i, j), as follows:

$(\delta B_t)^2 = \frac{(B_x \delta B_x)^2 + (B_y \delta B_y) + (B_z \delta B_z)^2}{{B_t}^2}.$

The errors in the quantities below follow similarly:

$$\overline{|\nabla B_h|} = \frac{1}{N}\sum_i \sqrt{\left(\frac{\partial B_{h_i}}{\partial x}\right)^2 + \left(\frac{\partial B_{h_i}}{\partial y}\right)^2} \text{ and } \overline{|\nabla B_z|} = \frac{1}{N}\sum_i \sqrt{\left(\frac{\partial B_{z_i}}{\partial x}\right)^2 + \left(\frac{\partial B_{z_i}}{\partial y}\right)^2}.$$

4 $\overline{J_z} = \frac{1}{N} \sum_i J_{z_i}$

The error in $\overline{J_z}$ is as follows: $\left(\delta \overline{J_z}\right)^2 = \frac{1}{N^2} \sum_i \left(\delta J_{z_i}\right)^2$.

There are two approaches one can take to calculate δJ_z: calculate the error terms assuming (i) an analytical form for the curl, i.e. $J_z = \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}$, or (ii) a numerical form for the curl, e.g. `0.5*(Bx[i,j+1] - Bx[i,j-1]) - 0.5*(By[i+1,j] - By[i-1,j])`. I am going to go with the numerical form.

Using a simple finite difference method, the derivatives are defined as follows:
$\frac{\partial B_x}{\partial y}$ =`0.5*(Bx[i,j+1] - Bx[i,j-1])` and $\frac{\partial B_y}{\partial x}$ =`0.5*(By[i+1,j] - By[i-1,j])`.

Thus, `Jz[i,j]` = `0.5*(Bx[i,j+1] - Bx[i,j-1] - By[i+1,j] + By[i-1,j])`.

Therefore, the error in in J_z is as follows:
$(\delta \mathtt{Jz[i,j]})^2 = \frac{1}{2(\Delta x)^2} \left[(\delta \mathtt{Bx[i,j+1]})^2 + (\delta \mathtt{Bx[i,j-1]})^2 + (\delta \mathtt{Bx[i+1,j]})^2 + (\delta \mathtt{Bx[i-1,j]})^2 \right]$.

5 $J_{z_{total}} = \sum_i C J_{z_i}$

The error in $J_{z_{total}}$ is $(\delta J_{z_{total}})^2 = \sum_i C^2 \left(\delta J_{z_i}\right)^2$.

6 $\alpha_{total} = C \frac{\sum_i J_{z_i} B_{z_i}}{\sum_i B_{z_i}^2}$

The general formula for the twist parameter, α, is simply $\alpha = C \frac{J_z}{B_z}$, where C is a constant. We've calculated a modified version from Hagino et al., which I'm explicitly calling α_{total} because we are dividing sums by sums; we are not strictly taking a mean, i.e. a moment of a distribution.

First, for purposes of simplicity, let's assume that there are only two pixels and α_{total} is reduced to:

$\alpha_{simplified} = C \frac{J_{z_1} B_{z_1} + J_{z_2} B_{z_2}}{B_{z_1}^2 + B_{z_2}^2}$.

The error in in $\alpha_{simplified}$ is as follows:
$(\delta \alpha_{simplified})^2 = \left(\frac{\partial \alpha_{simplified}}{\partial J_{z_1}} \delta J_{z_1} \right)^2 + \left(\frac{\partial \alpha_{simplified}}{\partial J_{z_2}} \delta J_{z_2} \right)^2 + \left(\frac{\partial \alpha_{simplified}}{\partial B_{z_1}} \delta B_{z_1} \right)^2 + \left(\frac{\partial \alpha_{simplified}}{\partial B_{z_2}} \delta B_{z_2} \right)^2$.

The partial derivatives are as follows:
$\frac{\partial \alpha_{simplified}}{\partial J_{z_1}} = C \frac{B_{z_1}}{B_{z_1}^2 + B_{z_2}^2}$, and similarly for J_{z_2}.

$\frac{\partial \alpha_{simplified}}{\partial B_{z_1}} = C\left(\frac{J_{z_1}}{B_{z_1}^2+B_{z_2}^2} - \frac{2B_{z_1}\left(J_{z_1}B_{z_1}+J_{z_2}B_{z_2}\right)}{\left(B_{z_1}^2+B_{z_2}^2\right)^2}\right)$, and similarly for B_{z_2}.

Therefore, the error in α_{total} is simply a generalization of the error in $\alpha_{simplified}$:

$$(\delta\alpha_{total})^2 = \sum_i \left[C\frac{B_{z_i}}{\sum_i B_{z_i}^2}\delta J_{z_i}\right]^2 + \sum_i \left[C\left(\frac{J_{z_i}}{\sum_i B_{z_i}^2} - \frac{2B_{z_i}\left(\sum_i J_{z_i}B_{z_i}\right)}{\left(\sum_i B_{z_i}^2\right)^2}\right)\delta B_{z_i}\right]^2.$$

7 $\overline{H_c} = \frac{1}{N}\sum_i CB_{z_i}J_{z_i}$

The error in in H_c for one pixel, (i, j), is as follows:

$(\delta H_c)^2 = \left(\frac{\partial H_c}{\partial B_z}\delta B_z\right)^2 + \left(\frac{\partial H_c}{\partial J_z}\delta J_z\right)^2$.
$(\delta H_c)^2 = (CJ_z\delta B_z)^2 + (CB_z\delta J_z)^2$.

We report the mean value of H_c, which means we must also determine the error in $\overline{H_c}$:
$\left(\delta\overline{H_c}\right)^2 = \frac{1}{N^2}\sum_i (\delta H_{c_i})^2$.

We also compute two similar quantities: $H_{c_{total}} = \sum_i |B_{z_i}J_{z_i}|$ and $H_{c_{abs}} = \left|\sum_i B_{z_i}J_{z_i}\right|$. The same error formula applies to both of these quantities, namely:

$(\delta H_{c_{total}})^2 = (\delta H_{c_{abs}})^2 = \sum_i (\delta H_{c_i})^2$.

8 $J_{z_{summed}} = C\left|\sum^{B_z^+} J_{z_i}\right| + C\left|\sum^{B_z^-} J_{z_i}\right|$

Because errors are always positive, the error in $J_{z_{summed}}$ is the same as the error in $\sum_i CJ_{z_i}$.

Therefore, the error in $J_{z_{summed}}$ is simply: $(J_{z_{summed}})^2 = \sum_i C^2 (\delta J_{z_i})^2$.

9 $\overline{\rho} = \frac{1}{N}\sum_i C\left(\vec{B}_i^o - \vec{B}_i^p\right)^2$

Since the z-component of the observed field is used to generate the potential field, $B_z^o = B_z^p$ and ρ at one pixel, (i, j), is reduced to: $\rho = C(B_x^o - B_x^p)^2 + C\left(B_y^o - B_y^p\right)^2$. We assume no error in the potential components of the field; thus our value of $\delta\overline{\rho}$ represents a lower bound. Another approach, which we do not take, is to assume that the order of $\delta\vec{B}^o = \delta\vec{B}^p$, which means that our value of $\delta\overline{\rho}$ would include some multiplicative factor. The error in ρ at one pixel, (i,j), is reduced to:

$(\delta\rho)^2 = \left(\frac{\partial\rho}{\partial B_x^o}\delta B_x^o\right)^2 + \left(\frac{\partial\rho}{\partial B_y^o}\delta B_y^o\right)^2.$

$(\delta\rho)^2 = \left[2C\left(B_x^o - B_x^p\right)\delta B_x^o\right]^2 + \left[2C\left(B_y^o - B_y^p\right)\delta B_y^o\right]^2.$

We report the mean value of ρ, which means we must also determine the error in $\overline{\rho}$:
$(\delta\overline{\rho})^2 = \frac{1}{N^2}\sum_i C^2\left(\delta\rho_i\right)^2.$

10 $\rho_{total} = \sum_i \left(\vec{B}_i^o - \vec{B}_i^p\right)^2 dA^2$

Following from the section above, the error in ρ_{total} is:

$(\delta\rho_{total})^2 = \sum_i \left[2\left(B_{x_i}^o - B_{x_i}^p\right)\delta B_{x_i}^o dA\right]^2 + \sum_i \left[2\left(B_{y_i}^o - B_{y_i}^p\right)\delta B_{y_i}^o dA\right]^2.$

11 $\overline{\chi} = \frac{1}{N}\sum_i \arccos\left(\frac{\vec{B}_i^o \cdot \vec{B}_i^p}{|B_i^o|\,|B_i^p|}\right)$

χ at one pixel, (i,j): $\chi = \arccos\left(\frac{B_x^oB_x^p + B_y^oB_y^p + B_z^oB_z^p}{\sqrt{B_x^{o2}+B_y^{o2}+B_z^{o2}}\sqrt{B_x^{p2}+B_y^{p2}+B_z^{p2}}}\right).$

Since $\delta B_x^p = \delta B_y^p = \delta B_z^p = 0$, the error in χ is as follows:
$(\delta\chi)^2 = \left(\frac{\partial\rho}{\partial B_x^o}\delta B_x^o\right)^2 + \left(\frac{\partial\rho}{\partial B_y^o}\delta B_y^o\right)^2 + \left(\frac{\partial\rho}{\partial B_z^o}\delta B_z^o\right)^2.$

Let's compute each term:

$$TERM1 \equiv \frac{\partial\rho}{\partial B_x^o}\delta B_x^o = \frac{B_x^oB_y^oB_y^p - B_y^{o2}B_x^p + B_z^o\left(B_x^oB_z^p - B_z^oB_x^p\right)}{\left(B_x^{o2}+B_y^{o2}+B_z^{o2}\right)^{3/2}\sqrt{B_x^{p2}+B_y^{p2}+B_z^{p2}}\sqrt{1-\frac{\left(B_x^oB_x^p+B_y^oB_y^p+B_z^oB_z^p\right)^2}{\left(B_x^{o2}+B_y^{o2}+B_z^{o2}\right)\left(B_x^{p2}+B_y^{p2}+B^pz^2\right)}}}\delta B_x^o.$$

$$TERM2 \equiv \frac{\partial\rho}{\partial B_y^o}\delta B_y^o = -\frac{B_x^{o2}B_y^p - B_x^oB_y^oB_x^p + B_z^o\left(B_z^oB_y^p - B_y^oB_z^p\right)}{\left(B_x^{o2}+B_y^{o2}+B_z^{o2}\right)^{3/2}\sqrt{B_x^{p2}+B_y^{p2}+B_z^{p2}}\sqrt{1-\frac{\left(B_x^oB_x^p+B_y^oB_y^p+B_z^oB_z^p\right)^2}{\left(B_x^{o2}+B_y^{o2}+B_z^{o2}\right)\left(B_x^{p2}+B_y^{p2}+B^pz^2\right)}}}\delta B_y^o.$$

$$TERM3 \equiv \frac{\partial\rho}{\partial B_x^o}\delta B_z^o = -\frac{B_x^{o2}B_z^p - B_x^oB_z^oB_x^p + B_y^o\left(B_y^oB_z^p - B_z^oB_y^p\right)}{\left(B_x^{o2}+B_y^{o2}+B_z^{o2}\right)^{3/2}\sqrt{B_x^{p2}+B_y^{p2}+B_z^{p2}}\sqrt{1-\frac{\left(B_x^oB_x^p+B_y^oB_y^p+B_z^oB_z^p\right)^2}{\left(B_x^{o2}+B_y^{o2}+B_z^{o2}\right)\left(B_x^{p2}+B_y^{p2}+B^pz^2\right)}}}\delta B_z^o.$$

Therefore, $(\delta\chi)^2 = TERM1^2 + TERM2^2 + TERM3^2$. We report the mean value of χ, which means we must also determine the error in $\overline{\chi}$:

$(\delta\overline{\chi})^2 = \frac{1}{N^2}\sum_i \left(\delta\chi_i\right)^2.$